

UNIVERSAL MOCK THETA FUNCTIONS AND TWO-VARIABLE HECKE-ROGERS IDENTITIES

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ABSTRACT. We obtain two-variable Hecke-Rogers identities for three universal mock theta functions. This implies that many of Ramanujan's mock theta functions, including all the third order functions, have a Hecke-Rogers-type double sum representation. We find new generating function identities for the Dyson rank function, the overpartition rank function, the M_2 -rank function and related spt-crank functions. Results are proved using the theory of basic hypergeometric functions.

1. INTRODUCTION

In this paper we obtain two-variable generalizations of the following Hecke-Rogers identities.

$$(1.1) \quad \prod_{n=1}^{\infty} (1 - q^n)^2 = \sum_{n=0}^{\infty} \sum_{m=-[n/2]}^{[n/2]} (-1)^{n+m} q^{\frac{1}{2}(n^2-3m^2) + \frac{1}{2}(n+m)},$$

$$(1.2) \quad \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n}) = \sum_{n=0}^{\infty} \sum_{m=-[n/2]}^{[n/2]} (-1)^{n+m} q^{\frac{1}{2}(n^2-2m^2) + \frac{1}{2}n},$$

$$(1.3) \quad \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n}) = \sum_{n=0}^{\infty} \sum_{m=-[n/3]}^{[n/3]} (-1)^n q^{\frac{1}{2}(n^2-8m^2) + \frac{1}{2}n},$$

$$(1.4) \quad \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n}) = \sum_{n=0}^{\infty} \sum_{m=-n}^n (-1)^n q^{\frac{1}{2}(2n^2-m^2) + \frac{1}{2}(2n+m)}.$$

Hecke [23] was the first to systematically consider identities of this type. Equation (1.1) was found by Hecke [23, Equation (7), p.425] but is originally due to L. J. Rogers [36, p.323]. Identities of this type arose in Kac and Petersen's [29] work on character

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formulas for infinite dimensional Lie algebras and string functions. Equation (1.3) is due to Kac and Petersen [29, final equation]. Andrews [4] derived (1.2), (1.3) using his constant term method. Bressoud [13] derived (1.2), (1.4) using q -Hermite polynomials.

Our generalizations of (1.1)–(1.4) are in terms of the universal mock theta functions

$$(1.5) \quad R(z, q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq; q)_n (z^{-1}q; q)_n},$$

$$(1.6) \quad H(z, q) = \sum_{n=0}^{\infty} \frac{(-1; q)_n q^{\frac{1}{2}n(n+1)}}{(zq; q)_n (z^{-1}q; q)_n},$$

$$(1.7) \quad K(z, q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{n^2}}{(zq^2; q^2)_n (z^{-1}q^2; q^2)_n}.$$

Here and throughout the paper we use the standard q -notation.

$$\begin{aligned} (a; q)_{\infty} &= \prod_{k=0}^{\infty} (1 - aq^k), \\ (a; q)_n &= \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}}, \\ (a_1, a_2, \dots, a_j; q)_{\infty} &= (a_1; q)_{\infty} (a_2; q)_{\infty} \dots (a_j; q)_{\infty}, \\ (a_1, a_2, \dots, a_j; q)_n &= (a_1; q)_n (a_2; q)_n \dots (a_j; q)_n. \end{aligned}$$

The functions (1.5)–(1.7) are called *universal mock theta* functions because Hickerson [24], [25] and Gordon and McIntosh [22] have shown that each of the classical mock theta functions may be expressed as specializations of these functions up to the addition of a modular form.

We note that the functions $R(z, q)$, $H(z, q)$ and $K(z, q)$ are generating functions for various rank-type functions. Let

$N(m, n)$ = the number of partitions of n with Dyson rank m ([18]),

$\overline{N}(m, n)$ = the number of overpartitions of n with Dyson rank m ([15]),

$N_2(m, n)$ = the number of partitions of n with distinct odd parts with M_2 -rank m ([11], [32]).

Then

$$(1.8) \quad \sum_{n=0}^{\infty} \sum_m N(m, n) z^m q^n = R(z, q),$$

$$(1.9) \quad \sum_{n=0}^{\infty} \sum_m \overline{N}(m, n) z^m q^n = H(z, q),$$

$$(1.10) \quad \sum_{n=0}^{\infty} \sum_m N2(m, n) (-1)^n z^m q^n = K(z, q),$$

so that

$$(1.11) \quad R(1, q) = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)},$$

$$(1.12) \quad H(1, q) = \prod_{n=1}^{\infty} \frac{(1 + q^n)}{(1 - q^n)} = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})}{(1 - q^n)^2},$$

$$(1.13) \quad K(1, q) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n-1})}{(1 - q^{2n})} = \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - q^{2n})^2},$$

$$(1.14) \quad K(1, -q) = \prod_{n=1}^{\infty} \frac{(1 + q^{2n-1})}{(1 - q^{2n})}.$$

We now collect our generalizations of (1.1)–(1.4) into

Theorem 1.1.

$$(1.15) \quad (zq)_{\infty} (z^{-1}q)_{\infty} (q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n} = (zq)_{\infty} (z^{-1}q)_{\infty} (q)_{\infty} R(z, q)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\sum_{j=0}^{[n/2]} (-1)^{n+j} (z^{n-3j} + z^{3j-n}) q^{\frac{1}{2}(n^2-3j^2) + \frac{1}{2}(n-j)} \right.$$

$$\left. + \sum_{j=1}^{[n/2]} (-1)^{n+j} (z^{n-3j+1} + z^{3j-n-1}) q^{\frac{1}{2}(n^2-3j^2) + \frac{1}{2}(n+j)} \right),$$

$$(1.16) \quad (1 + z)(zq)_{\infty} (z^{-1}q)_{\infty} (q)_{\infty} \sum_{n=0}^{\infty} \frac{(-1)_n q^{\frac{1}{2}n(n+1)}}{(zq)_n (z^{-1}q)_n} = (1 + z)(zq)_{\infty} (z^{-1}q)_{\infty} (q)_{\infty} H(z, q)$$

$$= \sum_{n=0}^{\infty} \sum_{|m| \leq [n/2]} (-1)^{n+m} (z^{n-2|m|+1} + z^{2|m|-n}) q^{\frac{1}{2}(n^2-2m^2) + \frac{1}{2}n}$$

$$(1.17) \quad = \sum_{n=0}^{\infty} \sum_{|m| \leq [n/3]} (-1)^n (z^{n+1-4|m|} + z^{4|m|-n}) q^{\frac{1}{2}(n^2-8m^2) + \frac{1}{2}n},$$

(1.18)

$$\begin{aligned}
& (zq^2; q^2)_\infty (z^{-1}q^2; q^2)_\infty (q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q; q^2)_n}{(zq^2; q^2)_n (z^{-1}q^2; q^2)_n} \\
&= (zq^2; q^2)_\infty (z^{-1}q^2; q^2)_\infty (q^2; q^2)_\infty K(z, q) \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n (-1)^n z^{m-n} q^{\frac{1}{2}(2n^2-m^2)+\frac{1}{2}(2n-m)} + \sum_{m=1}^n (-1)^n z^{n-m+1} q^{\frac{1}{2}(2n^2-m^2)+\frac{1}{2}(2n+m)} \right).
\end{aligned}$$

In view of (1.11)–(1.13) we see that the Hecke-Rogers identities (1.1)–(1.4) follow by putting $z = 1$ in (1.15)–(1.18).

Corollary 1.2. *Each of Ramanujan's third order mock theta functions has a Hecke-Rogers-type double sum representation.*

Remark 1.3. Previously the only such representations were known for the third order functions $\psi(q)$ (Andrews [6]), and $\phi(q)$, $\nu(q)$ (Mortenson [34]). We also note that Hickerson and Mortenson [27] have found Hecke-Rogers double sum representations for all the classical mock theta functions except those of third order.

Proof. We recall the universal mock theta function

$$(1.19) \quad g(x, q) := x^{-1} \left(-1 + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(x)_{n+1} (x^{-1}q)_n} \right).$$

It is well-known that each of Ramanujan's third mock theta functions $f(q)$, $\phi(q)$, $\psi(q)$, $\chi(q)$, $\omega(q)$, $\nu(q)$, $\rho(q)$ can be written solely in terms of $g(x, q)$. These expressions have been cataloged by Hickerson and Mortenson [26, (5.4)–(5.10)]. The result follows from Theorem 1.1 since

$$(1.20) \quad g(x, q) = x^{-1} \left(-1 + \frac{1}{1-x} R(x, q) \right).$$

□

A striking example is Ramanujan's third order mock theta function

$$(1.21) \quad f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2} = R(-1, q).$$

Putting $z = -1$ in (1.15) gives

$$(1.22) \quad f(q) = \frac{(q)_\infty}{(q^2; q^2)_\infty^2} \sum_{n=0}^{\infty} \sum_{m=-[n/2]}^{[n/2]} \operatorname{sgn}(m) q^{\frac{1}{2}(n^2-3m^2)+\frac{1}{2}(n-m)},$$

where $\text{sgn}(m) = 1$ if $m \geq 0$ and otherwise $\text{sgn}(m) = -1$. We may rewrite this identity as

$$(1.23) \quad \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} f(q) = \sum_{n=0}^{\infty} \sum_{m=-[n/2]}^{[n/2]} \text{sgn}(m) q^{\frac{1}{2}(n^2-3m^2)+\frac{1}{2}(n-m)}.$$

This gives a recurrence for the coefficients of the q -series of $f(q)$. A similar but different result was found by Imamoğlu, Raum and Richter [28, Theorem 1.1] using the method of holomorphic projection applied to harmonic weak Maass forms.

By putting $z = -1$ in (1.18) we obtain a similar identity for the second order mock theta function

$$(1.24) \quad \mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q; q^2)_n}{(-q^2; q^2)_n^2} = K(-1, q),$$

namely

$$(1.25) \quad \mu(q) = \frac{(q^2; q^2)_{\infty}}{(q^4; q^4)_{\infty}^2} \sum_{n=0}^{\infty} \sum_{m=-n}^n \text{sgn}(m) (-1)^m q^{\frac{1}{2}(2n^2-m^2)+\frac{1}{2}(2n-m)},$$

or

$$(1.26) \quad \sum_{n=0}^{\infty} q^{n(n+1)} \mu(q) = \sum_{n=0}^{\infty} \sum_{m=-n}^n \text{sgn}(m) (-1)^m q^{\frac{1}{2}(2n^2-m^2)+\frac{1}{2}(2n-m)},$$

This confirms an identity found earlier by Hickerson and Mortenson [27]. For completeness we examine (1.16) near $z = -1$. We divide both sides by $(1+z)$, let $z \rightarrow -1$ and simplify to obtain

$$(1.27) \quad \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} \sum_{n=1}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}}{(-q; q)_n (1+q^n)} = \sum_{n=1}^{\infty} \sum_{m=0}^{[n/2]} (-1)^m (2n-4m+1-\delta_{m,0}(n+1)) q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n},$$

where $\delta_{m,0} = 1$ if $m = 0$ and $\delta_{m,0} = 0$ otherwise.

2. GENESIS

We now describe how we were led to our two-variable Hecke-Rogers identities. It began with a new identity for the Andrews [5] spt -function. Let $\text{spt}(n)$ denote the number of smallest parts in the partitions of n . Then

$$(2.1) \quad \begin{aligned} \sum_{n=1}^{\infty} \text{spt}(n) q^n &= \sum_{n=1}^{\infty} (q^n + 2q^{2n} + 3q^{3n} + \cdots) \frac{1}{(q^{n+1}; q)_{\infty}} \\ &= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q^n; q)_{\infty}} \end{aligned}$$

$$= \frac{1}{(q; q)_\infty} \sum_{n=1}^{\infty} \frac{q^n (q; q)_{n-1}}{(1 - q^n)}.$$

Andrews [5, Theorem 4] found that

$$(2.2) \quad \sum_{n=1}^{\infty} \text{spt}(n) q^n = \frac{1}{(q; q)_\infty} \left(\sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(3n+1)} (1 + q^n)}{(1 - q^n)^2} \right) \\ = \frac{1}{(q; q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\frac{1}{2}n(n+1)} (1 - q^{n^2}) (1 + q^n)}{(1 - q^n)^2},$$

by letting $z = 1$ in [9, Theorem 2.4]. This generating function identity provides an efficient method for calculating the spt-coefficients. We find that

$$\sum_{n=1}^{\infty} \text{spt}(n) q^n = q + 3q^2 + 5q^3 + 10q^4 + 14q^5 + 26q^6 + \cdots \\ \cdots + 600656570957882248155746472836274 q^{1000} + \cdots$$

We tried multiplying by different powers of $\prod_{n=1}^{\infty} (1 - q^n)$ and stumbled upon

$$(2.3) \quad \prod_{n=1}^{\infty} (1 - q^n)^3 \sum_{n=1}^{\infty} \text{spt}(n) q^n = q - 4q^3 - q^5 + 9q^6 + q^8 + 4q^9 - 16q^{10} - 4q^{13} + \cdots \\ \cdots - 1936q^{990} - 900q^{995} - 49q^{996} - 705q^{1000} + \cdots,$$

which suggests something is going on (705 = 961 - 256 is the first non-square!). The final result is given below in Corollary 2.9. If we let

$$\sum_{n=1}^{\infty} a(n) q^n = \prod_{n=1}^{\infty} (1 - q^n)^3 \sum_{n=1}^{\infty} \text{spt}(n) q^n,$$

then we find

$$\sum_{n=0}^{\infty} a(5n+2) q^n = -25q^5 + 100q^{15} + 25q^{25} - 225q^{30} - 25q^{40} - 100q^{45} + 400q^{50} + \cdots,$$

and one is led to conjecture that

$$(2.4) \quad a(5n+2) = -25a(n/5),$$

for $n \geq 0$. This is quite surprising since the generating function for $\text{spt}(n)$ is not a modular form but a quasi-mock modular form from the work of Bringmann [14]. A similar behaviour occurs for any prime $\ell \equiv \pm 5 \pmod{12}$. See Corollary 2.10 below for the general result.

The idea is to find a z -analog of (2.3). In [9] we found a nice z -analog of the generating function (2.1)

$$\begin{aligned} S(z, q) &= \sum_{n=1}^{\infty} \sum_m N_S(m, n) z^m q^n \\ &= \sum_{n=1}^{\infty} \frac{q^n (q^{n+1}; q)_{\infty}}{(zq^n; q)_{\infty} (z^{-1}q^n; q)_{\infty}}. \end{aligned}$$

We note that

$$S(1, q) = \sum_{n=1}^{\infty} \text{spt}(n) q^n.$$

From [9, (2.5)], [10, (3.23)] we have the following generating function identities.

$$\begin{aligned} S(z, q) &= \frac{1}{(q)_{\infty}} \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2}}{(1 - zq^n)(1 - z^{-1}q^n)} - \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{n(3n+1)/2}}{(1 - zq^n)(1 - z^{-1}q^n)} \right), \\ (2.5) \quad &= \frac{1}{(zq)_{\infty}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2}}{(q)_n (1 - z^{-1}q^n)} \left(\frac{z^n - 1}{z - 1} \right). \end{aligned}$$

We find that

$$(2.6)$$

$$\begin{aligned} &(z; q)_{\infty} (z^{-1}; q)_{\infty} (q; q)_{\infty} S(z, q) \\ &= \frac{(-z^2 + 2z - 1)q}{z} + \frac{(z^4 - 2z^2 + 1)q^3}{z^2} + \frac{(z^2 - 2z + 1)q^5}{z} + \frac{(-z^6 + 2z^3 - 1)q^6}{z^3} \\ &+ \frac{(-z^2 + 2z - 1)q^8}{z} + \frac{(-z^4 + 2z^2 - 1)q^9}{z^2} + \frac{(z^8 - 2z^4 + 1)q^{10}}{z^4} + \dots, \end{aligned}$$

and we are clearly on the right track. We are eventually led to conjecture

Theorem 2.1.

$$(2.7)$$

$$(z)_{\infty} (z^{-1})_{\infty} (q)_{\infty} S(z, q) = \sum_{n=0}^{\infty} \sum_{m=0}^n (1 - z^{\frac{1}{2}(n-m)})^2 z^{\frac{1}{2}(m-n)} \left(\frac{-4}{n} \right) \left(\frac{12}{m} \right) q^{\frac{1}{12}(\frac{3n^2 - m^2}{2} - 1)},$$

where (\cdot) is the Kronecker symbol.

In this section we prove Theorem 2.1 using the theory of basic hypergeometric functions including the method of Bailey pairs. We start with equation (2.5). We will need the following identity [10, p.216]:

$$(2.8) \quad \frac{(q)_{\infty}}{(z^{-1}q)_{\infty}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (1 - z^{-1})}{(1 - z^{-1}q^n)(q)_n}.$$

As noted in [10, Theorem 3.5] this is related to the spt-like function due to Fokkink, Fokkink and Wang [19]. See also Andrews [5, p.134].

A pair of sequences $(\alpha_n(a, q), \beta_n(a, q))$ is called a *Bailey pair* with parameters (a, q) if

$$(2.9) \quad \beta_n(a, q) = \sum_{r=0}^n \frac{\alpha_r(a, q)}{(q; q)_{n-r} (aq; q)_{n+r}}$$

for all $n \geq 0$. We will need

Lemma 2.2 (Bailey's Lemma). *Suppose $(\alpha_n(a, q), \beta_n(a, q))$ is a Bailey pair with parameters (a, q) . Then $(\alpha'_n(a, q), \beta'_n(a, q))$ is another Bailey pair with parameters (a, q) , where*

$$\begin{aligned} \alpha'_n(a, q) &= \frac{(\rho_1; q)_n (\rho_2; q)_n \left(\frac{aq}{\rho_1 \rho_2}\right)^n}{\left(\frac{aq}{\rho_1}; q\right)_n \left(\frac{aq}{\rho_2}; q\right)_n} \alpha_n(a, q), \\ \beta'_n(a, q) &= \sum_{j=0}^n \frac{(\rho_1; q)_j (\rho_2; q)_j \left(\frac{aq}{\rho_1 \rho_2}\right)^j}{(q; q)_{n-j} \left(\frac{aq}{\rho_1}; q\right)_n \left(\frac{aq}{\rho_2}; q\right)_n} \beta_j(a, q). \end{aligned}$$

By letting $\rho_1, \rho_2 \rightarrow \infty$ we obtain

Corollary 2.3 (Limiting Form of Bailey's Lemma). *Suppose $(\alpha_n(a, q), \beta_n(a, q))$ is a Bailey pair with parameters (a, q) . Then $(\alpha'_n(a, q), \beta'_n(a, q))$ is another Bailey pair with parameters (a, q) , where*

$$\begin{aligned} \alpha'_n(a, q) &= a^n q^{n^2} \alpha_n(a, q) \\ \beta'_n(a, q) &= \sum_{j=0}^n \frac{a^j q^{j^2} \beta_j(a, q)}{(q)_{n-j}}. \end{aligned}$$

By letting $n \rightarrow \infty$ and using (2.9) we obtain

$$(2.10) \quad \sum_{j=0}^{\infty} a^j q^{j^2} \beta_j = \frac{1}{(aq; q)_{\infty}} \sum_{r=0}^{\infty} a^r q^{r^2} \alpha_r,$$

for any Bailey pair (α_n, β_n) with parameters (a, q) .

Proposition 2.4. *The following form a Bailey pair.*

$$(2.11) \quad \alpha_n = \begin{cases} 1, & n = 0, \\ q^{n^2} (a^n q^n - a^{n-1} q^{-n}), & n \geq 1, \end{cases} \quad \beta_n = \frac{q^n}{(q)_n (aq; q)_n}.$$

Proof. In [37, Eqn.(4.1), p.468] we let $c = q$ and $d \rightarrow \infty$ to obtain

$$(2.12) \quad \sum_{r=0}^n \frac{(1 - aq^{2r}) q^{r^2-r} a^r}{(a)_{n+r+1} (q)_{n-r}} = \frac{1}{(q)_n (a)_n}.$$

We have

$$\begin{aligned}
\sum_{r=1}^n \frac{(1-aq^{2r})q^{r^2-r}a^r}{(aq)_{n+r}(q)_{n-r}} &= (1-a) \left(\frac{1}{(q)_n(a)_n} - \frac{(1-a)}{(a)_{n+1}(q)_n} \right), \\
\sum_{r=1}^n \frac{q^{r^2}(a^r q^r - a^{r-1} q^{-r})}{(aq)_{n+r}(q)_{n-r}} &= (1-a^{-1}) \left(\frac{1}{(q)_n(a)_n} - \frac{1}{(aq)_n(q)_n} \right), \\
\frac{1}{(aq)_n(q)_n} + \sum_{r=1}^n \frac{q^{r^2}(a^r q^r - a^{r-1} q^{-r})}{(aq)_{n+r}(q)_{n-r}} &= -a^{-1} \frac{(1-a)}{(q)_n(a)_n} + \frac{a^{-1}}{(aq)_n(q)_n} \\
&= a^{-1} \left(\frac{-1}{(a)_n(aq)_{n-1}} + \frac{1}{(aq)_n(q)_n} \right) = a^{-1} \frac{-(1-aq^n) + 1}{(a)_n(aq)_n} = \frac{q^n}{(aq)_n(q)_n},
\end{aligned}$$

and the result follows. \square

Proposition 2.5. *For any nonnegative integer n we have*

$$(2.13) \quad \sum_{j=0}^n \frac{a^j q^{j^2+j}}{(q)_{n-j}(q)_j(aq)_j} = \sum_{j=0}^n \frac{(-1)^j a^j q^{j(j+1)/2}}{(q)_{n-j}(aq)_n}.$$

Proof. We need [21, p.241, (III.7)]:

$${}_2\phi_1 \left(\begin{matrix} q^{-n}, & \beta; & q, & z \\ & \gamma \end{matrix} \right) = \frac{(\gamma/\beta)_n}{(\gamma)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, & \beta, & \frac{\beta z q^{-n}}{\gamma}; & q, & q \\ & \frac{\beta q^{1-n}}{\gamma}, & 0 \end{matrix} \right).$$

We make the substitutions $\gamma = aq$, $\beta = c^{-1}q$, $z = acq^{n+1}$, so that $\frac{\beta z q^{-n}}{\gamma} = q$ and

$${}_2\phi_1 \left(\begin{matrix} q^{-n}, & c^{-1}q; & q, & acq^{n+1} \\ & aq \end{matrix} \right) = \frac{(ac)_n}{(aq)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, & c^{-1}q, & q; & q, & q \\ & \frac{c^{-1}q^{1-n}}{a}, & 0 \end{matrix} \right).$$

Letting $c \rightarrow 0^+$ and then dividing both sides by $(q)_n$ we obtain

$$\sum_{j=0}^n \frac{(q^{-n}; q)_j (-1)^j a^j q^{j(j+3)/2+nj}}{(q)_n(q)_j(aq)_j} = \sum_{j=0}^n \frac{(q^{-n}; q)_j a^j q^{j(1+n)}}{(q)_n(aq)_n}.$$

The result follows by [21, p.233, (I.12)]. \square

Proposition 2.6.

(2.14)

$$\sum_{m=0}^{\infty} \sum_{0 \leq k < m/3} (-1)^{m+k} z^{m-3k} q^{\frac{1}{2}(m^2-3k^2) + \frac{1}{2}(m-k)} = \sum_{m=0}^{\infty} \sum_{m/3 < k \leq m/2} (-1)^{m+k} z^{-m+3k} q^{\frac{1}{2}(m^2-3k^2) + \frac{1}{2}(m-k)},$$

(2.15)

$$\sum_{m=0}^{\infty} \sum_{1 \leq k < (m+1)/3} (-1)^{m+k} z^{m-3k+1} q^{\frac{1}{2}(m^2-3k^2) + \frac{1}{2}(m+k)} = \sum_{m=0}^{\infty} \sum_{(m+1)/3 < k \leq m/2} (-1)^{m+k} z^{-m+3k-1} q^{\frac{1}{2}(m^2-3k^2) + \frac{1}{2}(m+k)}.$$

Proof. We show (2.14) by showing that the coefficient of z^j agree on both sides for $j \geq 1$. On the left side this occurs when $m = 3k + j$, where $k \geq 0$. We have

$$\text{Coefficient of } z^j \text{ in LHS(2.14)} = \sum_{k \geq 0} (-1)^j q^{3k^2 + 3kj + \frac{1}{2}j(j+1) + k}.$$

On the right side we need $m = 3k - j$, and $2k \leq m$ so that $k \geq j$, and we have

$$\begin{aligned} \text{Coefficient of } z^j \text{ in RHS(2.14)} &= \sum_{k \geq j} (-1)^j q^{3k^2 - 3kj + \frac{1}{2}j(j-1) + k} \\ &= \sum_{k \geq 0} (-1)^j q^{3k^2 + 3kj + \frac{1}{2}j(j+1) + k} \\ &= \text{Coefficient of } z^j \text{ in LHS(2.14)}, \end{aligned}$$

by replacing k by $k + j$ in the first summation. This proves (2.14). The proof of (2.15) is similar. \square

This proposition leads to a new version of the Hecke-Rogers identity (1.1).

Corollary 2.7.

$$\begin{aligned} \prod_{n=1}^{\infty} (1 - q^n)^2 &= 2 \sum_{m=0}^{\infty} \left(\sum_{0 \leq k < m/3} (-1)^{m+k} q^{\frac{1}{2}(m^2 - 3k^2) + \frac{1}{2}(m-k)} \right. \\ &\quad \left. + \sum_{1 \leq k < (m+1)/3} (-1)^{m+k} q^{\frac{1}{2}(m^2 - 3k^2) + \frac{1}{2}(m+k)} \right) + \sum_{m=0}^{\infty} q^{3m^2+m} - \sum_{m=1}^{\infty} q^{3m^2-m}. \end{aligned}$$

Remark 2.8. A finite form of this result was found earlier by the author and Alex Berkovich [12].

Proof. From (1.1) we have

$$\prod_{n=1}^{\infty} (1 - q^n)^2 = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{[m/2]} (-1)^{m+k} q^{\frac{1}{2}(m^2 - 3k^2) + \frac{1}{2}(m-k)} + \sum_{k=1}^{[m/2]} (-1)^{m+k} q^{\frac{1}{2}(m^2 - 3k^2) + \frac{1}{2}(m+k)} \right).$$

The result follows by splitting each of these sums using (2.14)–(2.15) with $z = 1$. \square

We are now ready to prove Theorem 2.1. First we write (2.7) in the following equivalent form

$$\begin{aligned} (2.16) \quad &(z)_{\infty} (z^{-1})_{\infty} (q)_{\infty} S(z, q) \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{[n/3]} (-1)^{n+j} (z^{n-3j} - 2 + z^{3j-n}) q^{\frac{1}{2}(n^2 - 3j^2) + \frac{1}{2}(n-j)} \right) \end{aligned}$$

$$+ \sum_{j=1}^{\lfloor n/3 \rfloor} (-1)^{n+j} (z^{n-3j+1} - 2 + z^{3j-n-1}) q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)} \Bigg).$$

We will prove (2.7) by showing that the coefficient of z^k on both sides agree for each k . Since both sides are symmetric in z , z^{-1} we may assume $k \geq 0$. From (2.5) we have

$$\begin{aligned} (z)_\infty (z^{-1})_\infty (q)_\infty S(z, q) &= (z^{-1})_\infty (q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2} (1 - z^n)}{(q)_n (1 - z^{-1} q^n)} \\ (2.17) \quad &= -(q)_\infty^2 + (z^{-1})_\infty (q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n z^n q^{n(n+1)/2}}{(q)_n (1 - z^{-1} q^n)}, \end{aligned}$$

by (2.8). We now calculate the coefficient of z^k in the Laurent series of

$$(2.18) \quad F(z, q) = (z^{-1})_\infty \sum_{n=0}^{\infty} \frac{(-1)^n z^n q^{n(n+1)/2}}{(q)_n (1 - z^{-1} q^n)}.$$

By Andrews [3, Eq.(2.2.6)] we have

$$(2.19) \quad F(z, q) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{-n} q^{\frac{1}{2}n(n-1)}}{(q)_n} \sum_{m=0}^{\infty} \frac{(-1)^m z^m q^{m(m+1)/2}}{(q)_m} \sum_{N=0}^{\infty} z^{-N} q^{mN}.$$

The coefficient of z^k in (2.19) arises when $-n + m - N = k$. So we let $n = j - N$, $m = j + k$ where $j \geq N \geq 0$ and we find that

$$\begin{aligned} [z^k] F(z, q) &= (-1)^k \sum_{N=0}^{\infty} \sum_{j=N}^{\infty} \frac{(-1)^N q^{j^2+jk+\frac{1}{2}N(N+1)+Nk+\frac{1}{2}k(k+1)}}{(q)_{j+k} (q)_{j-N}} \\ (2.20) \quad &= (-1)^k q^{\frac{1}{2}k(k+1)} \sum_{j=0}^{\infty} \left(\sum_{N=0}^j \frac{(-1)^N q^{\frac{1}{2}N(N+1)+Nk}}{(q)_{j+k} (q)_{j-N}} \right) q^{j^2+jk}. \end{aligned}$$

We now apply the Limiting Form of Bailey's Lemma to the Bailey pair (2.11) to obtain the following Bailey pair with parameters (a, q) :

$$\begin{aligned} \alpha'_n &= a^n q^{n^2} \alpha_n \\ &= \begin{cases} 1, & n = 0, \\ q^{2n^2} (a^{2n} q^n - a^{2n-1} q^{-n}), & n \geq 1, \end{cases} \\ \beta'_n &= \sum_{j=0}^n \frac{a^j q^{j^2} \beta_j}{(q)_{n-j}} \\ &= \sum_{j=0}^n \frac{a^j q^{j^2+j}}{(q)_{n-j} (q)_j (aq)_j} \end{aligned}$$

$$= \sum_{j=0}^n \frac{(-1)^j a^j q^{j(j+1)/2}}{(q)_{n-j} (aq)_n},$$

by (2.13). Now using this Bailey pair in (2.10) with $a = q^k$ we have

$$(q)_k \sum_{j=0}^{\infty} \left(\sum_{n=0}^j \frac{(-1)^n q^{\frac{1}{2}n(n+1)+nk}}{(q)_{j+k} (q)_{j-n}} \right) q^{j^2+jk} = \frac{1}{(q^{k+1}; q)_{\infty}} \left(1 + \sum_{r=1}^{\infty} q^{3r^2+rk} (q^{2rk+r} - q^{2rk-k-r}) \right),$$

where we have used the fact that

$$(2.21) \quad (q^{k+1}; q)_j = \frac{(q)_{j+k}}{(q)_k}.$$

Thus we have

$$(2.22) \quad \sum_{j=0}^{\infty} \left(\sum_{n=0}^j \frac{(-1)^n q^{\frac{1}{2}n(n+1)+nk}}{(q)_{j+k} (q)_{j-n}} \right) q^{j^2+jk} = \frac{1}{(q; q)_{\infty}} \left(\sum_{r=0}^{\infty} q^{3r^2+3rk+r} - \sum_{r=1}^{\infty} q^{3r^2+3rk-r-k} \right).$$

We are now ready to show that the coefficient of z^k on both sides of equation (2.16) agree for all $k \geq 0$.

Case 1. $k = 0$. By (2.17), (2.18), (2.20), (2.22) we have

$$\begin{aligned} [z^0] \text{LHS}((2.16)) &= -(q)_{\infty}^2 + (q)_{\infty} \sum_{j=0}^{\infty} \left(\sum_{n=0}^j \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{(q)_j (q)_{j-n}} \right) q^{j^2} \\ &= -(q)_{\infty}^2 + \sum_{r=0}^{\infty} q^{3r^2+r} - \sum_{r=1}^{\infty} q^{3r^2-r} \\ &= -2 \sum_{m=0}^{\infty} \left(\sum_{0 \leq k < m/3} (-1)^{m+k} q^{\frac{1}{2}(m^2-3k^2)+\frac{1}{2}(m-k)} \right. \\ &\quad \left. + \sum_{1 \leq k < (m+1)/3} (-1)^{m+k} q^{\frac{1}{2}(m^2-3k^2)+\frac{1}{2}(m+k)} \right) \quad (\text{by Corollary 2.7}) \\ &= [z^0] \text{RHS}((2.16)). \end{aligned}$$

Case 2. $k \geq 1$. By (2.17), (2.18), (2.20), (2.22) we have

$$\begin{aligned} [z^k] \text{LHS}((2.16)) &= (q)_{\infty} (-1)^k q^{\frac{1}{2}k(k+1)} \sum_{j=0}^{\infty} \left(\sum_{n=0}^j \frac{(-1)^n q^{\frac{1}{2}n(n+1)+nk}}{(q)_{j+k} (q)_{j-n}} \right) q^{j^2+jk} \\ &= (-1)^k q^{\frac{1}{2}k(k+1)} \left(\sum_{r=0}^{\infty} q^{3r^2+3rk+r} - \sum_{r=1}^{\infty} q^{3r^2+3rk-r-k} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{\infty} (-1)^k q^{3j^2+3jk+j+\frac{1}{2}k(k+1)} + \sum_{j=1}^{\infty} (-1)^{k-1} q^{3j^2+3jk-j+\frac{1}{2}k(k-1)} \\
&= [z^k] \text{RHS}((2.16)).
\end{aligned}$$

This completes the proof of (2.7).

If we divide both sides of (2.7) by $(1-z)(1-z^{-1})$ and let $z \rightarrow 1$ we obtain

Corollary 2.9.

$$(2.23) \quad \prod_{n=1}^{\infty} (1-q^n)^3 \sum_{n=1}^{\infty} \text{spt}(n) q^n = - \sum_{n=0}^{\infty} \sum_{m=0}^n \left(\frac{n-m}{2} \right)^2 \left(\frac{-4}{n} \right) \left(\frac{12}{m} \right) q^{\frac{1}{12}(\frac{3n^2-m^2}{2}-1)}.$$

Define the function $\alpha(n)$ by

$$(2.24) \quad \sum_{n=1}^{\infty} \alpha(n) q^n = \prod_{n=1}^{\infty} (1-q^{12n})^3 \sum_{n=1}^{\infty} \text{spt}(n) q^{12n+1},$$

so that $\alpha(n) = 0$ if n is not a positive integer congruent to 1 (mod 12). We have

Corollary 2.10. *Suppose $\ell \equiv \pm 5 \pmod{12}$ is prime. Then*

$$\begin{aligned}
\alpha(\ell n) + \ell^2 \alpha(n/\ell) &= 0, & \text{if } \ell \equiv 5 \pmod{12}, \\
\alpha(\ell n) - \ell^2 \alpha(n/\ell) &= 0, & \text{if } \ell \equiv -5 \pmod{12}.
\end{aligned}$$

Proof. From (2.23) we have

$$(2.25) \quad \sum_{n=1}^{\infty} \alpha(n) q^n = - \sum_{n=0}^{\infty} \sum_{m=0}^n \left(\frac{n-m}{2} \right)^2 \left(\frac{-4}{n} \right) \left(\frac{12}{m} \right) q^{\frac{3n^2-m^2}{2}}.$$

Suppose ℓ is prime and $\ell \equiv \pm 5 \pmod{12}$. Then we observe that

$$3n^2 - j^2 \equiv 0 \pmod{\ell} \quad \text{if and only if} \quad n \equiv j \equiv 0 \pmod{\ell},$$

since 3 is quadratic nonresidue mod ℓ . Hence

$$\alpha(\ell n) = \left(\frac{-4}{\ell} \right) \left(\frac{12}{\ell} \right) \ell^2 \alpha(n/\ell),$$

which gives the result. □

3. A TWO-VARIABLE HECKE-ROGERS IDENTITY FOR THE DYSON RANK FUNCTION

In this section we prove (1.15). We use the fact that spt-crank function $S(z, q)$ can be written in terms of the Dyson rank function. By [9, Corollary 2.5] we have

$$(3.1) \quad S(z, q) = \frac{-1}{(1-z)(1-z^{-1})} \left(\frac{(q)_{\infty}}{(zq)_{\infty}(z^{-1}q)_{\infty}} - R(z, q) \right),$$

so that

$$(3.2) \quad (zq)_\infty (z^{-1}q)_\infty (q)_\infty R(z, q) = (z)_\infty (z^{-1})_\infty (q)_\infty S(z, q) + (q)_\infty^2.$$

By (2.16) and Proposition 2.6 we have

$$\begin{aligned} & (zq)_\infty (z^{-1}q)_\infty (q)_\infty R(z, q) \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{[n/3]} (-1)^{n+j} (z^{n-3j} - 2 + z^{3j-n}) q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)} \right) \\ & \quad + \sum_{j=1}^{[n/3]} (-1)^{n+j} (z^{n-3j+1} - 2 + z^{3j-n-1}) q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)} \right) + (q)_\infty^2 \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left(\sum_{j=0}^{[n/2]} (-1)^{n+j} (z^{n-3j} - 2 + z^{3j-n}) q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)} \right) \\ & \quad + \sum_{j=1}^{[n/2]} (-1)^{n+j} (z^{n-3j+1} - 2 + z^{3j-n-1}) q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)} \right) + (q)_\infty^2 \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left(\sum_{j=0}^{[n/2]} (-1)^{n+j} (z^{n-3j} + z^{3j-n}) q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)} \right) \\ & \quad + \sum_{j=1}^{[n/2]} (-1)^{n+j} (z^{n-3j+1} + z^{3j-n-1}) q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)} \right), \end{aligned}$$

by (1.1). This completes the proof of (1.15).

4. TWO-VARIABLE HECKE-ROGERS IDENTITIES FOR THE OVERPARTITION RANK FUNCTION

In this section we prove (1.16) and (1.17). First we prove (1.16). The other equation (1.17) we will follow from a transformation of Milne [33]. We need to prove some q -hypergeometric identities.

Proposition 4.1.

$$(4.1) \quad (1+z)(z; q)_n (z^{-1}; q)_n = \sum_{j=-n}^{n+1} (-1)^{j+1} \frac{(q)_{2n}}{(q)_{n+j} (q)_{n-j+1}} (1 - q^{2j-1}) z^j q^{\frac{1}{2}j(j-3)+1}.$$

Proof. The proof follows from the well-known finite form of the Jacobi triple product identity [3, p.49]

$$(4.2) \quad (z; q)_n (z^{-1}q; q)_n = \sum_{j=-n}^n (-1)^j z^j q^{\frac{1}{2}j(j-1)} \begin{bmatrix} 2n \\ n+j \end{bmatrix}_q,$$

by a lengthy but straightforward calculation. \square

The proofs of the following proposition and corollary are similar to the proofs of some identities in Chapter 9 of Andrews and Berndt's Volume I of Ramanujan's Lost Notebook [7].

Proposition 4.2.

$$(4.3) \quad (q)_\infty (zq; q^2)_\infty \sum_{n=0}^{\infty} \frac{(z; q^2)_n}{(zq; q)_n (q)_n} q^n = 1 + 2 \sum_{m=1}^{\infty} (-1)^m z^m q^{m^2}.$$

Proof. In this proof we will need the Rogers-Fine identity [7, (9.1.1)]

$$(4.4) \quad \sum_{n=0}^{\infty} \frac{(\alpha; q)_n}{(\beta; q)_n} \tau^n = \sum_{n=0}^{\infty} \frac{(\alpha; q)_n (\alpha\tau q/\beta; q)_n \beta^n \tau^n q^{n^2-n} (1 - \alpha\tau q^{2n})}{(\beta; q)_n (\tau; q)_{n+1}}.$$

We will prove (4.3) with z replaced by z^2 . Applying Heine's transformation [21, (III.1)] with $a = z$, $b = -z$, $c = z^2q$ and $z \mapsto q$ we obtain

$$(4.5) \quad \begin{aligned} (q)_\infty (z^2q; q^2)_\infty \sum_{n=0}^{\infty} \frac{(z^2; q^2)_n}{(z^2q; q)_n (q)_n} q^n &= (q)_\infty (z^2q; q^2)_\infty \sum_{n=0}^{\infty} \frac{(z; q)_n (-z; q)_n}{(z^2q; q)_n (q)_n} q^n \\ &= (q)_\infty (z^2q; q^2)_\infty \frac{(-z; q)_\infty (zq; q)_\infty}{(z^2q; q)_\infty (q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-zq; q)_n}{(zq; q)_n} (-z)^n \\ &= \sum_{n=0}^{\infty} \frac{(-z; q)_{n+1}}{(zq; q)_n} (-z)^n. \end{aligned}$$

In (4.4) we let $\alpha = -zq$, $\beta = zq$, and $\tau = -z$ to obtain

$$\sum_{n=0}^{\infty} \frac{(-zq; q)_n}{(zq; q)_n} (-z)^n = \sum_{n=0}^{\infty} \frac{(-zq; q)_n (zq; q)_n (-1)^n z^{2n} (1 - z^2 q^{2n+1}) q^{n^2}}{(zq; q)_n (-z; q)_{n+1}}$$

so that

$$(4.6) \quad \begin{aligned} \sum_{n=0}^{\infty} \frac{(-z; q)_{n+1}}{(zq; q)_n} (-z)^n &= \sum_{n=0}^{\infty} (-1)^n z^{2n} (1 - z^2 q^{2n+1}) q^{n^2} \\ &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n z^{2n} q^{n^2}. \end{aligned}$$

The result follows from (4.5) and (4.6) by replacing z^2 by z . \square

Proposition 4.2 gives some nice false theta function identities.

Corollary 4.3.

$$(4.7) \quad \sum_{n=0}^{\infty} \frac{(-z; q)_{n+1}}{(zq; q)_n} (-z)^n = 1 + 2 \sum_{n=1}^{\infty} (-1)^n z^{2n} q^{n^2}$$

$$(4.8) \quad \sum_{n=0}^{\infty} \frac{(z; q^2)_{n+1} (q; q^2)_n}{(-zq; q)_{2n+1}} z^n = 1 + 2 \sum_{j=1}^{\infty} (-1)^j z^j q^{j^2}.$$

Proof. Equation (4.7) is (4.6). To prove (4.8) we need the following transformation due to Andrews [2, p.67]

$$(4.9) \quad \sum_{n=0}^{\infty} \frac{(a; q^2)_n (b; q)_{2n}}{(q^2; q^2)_n (c; q)_{2n}} t^n = \frac{(b; q)_{\infty} (at; q^2)_{\infty}}{(c; q)_{\infty} (t; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(c/b; q)_n (t; q^2)_n}{(q; q)_n (at; q^2)_n} b^n.$$

We let $b = q$, $c = -zq^2$, $t = z$, and $a = zq^2$ in (4.9) to obtain

$$(4.10) \quad \begin{aligned} \sum_{n=0}^{\infty} \frac{(zq^2; q^2)_n (q; q)_{2n}}{(q^2; q^2)_n (-zq^2; q)_{2n}} z^n &= \frac{(q; q)_{\infty} (z^2 q^2; q^2)_{\infty}}{(-zq^2; q)_{\infty} (z; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(-zq; q)_n (z; q^2)_n}{(q; q)_n (z^2 q^2; q^2)_n} b^n, \\ \sum_{n=0}^{\infty} \frac{(zq^2; q^2)_n (q; q^2)_n}{(-zq^2; q)_{2n}} z^n &= \frac{1+zq}{1-z} \frac{(q; q)_{\infty} (z^2 q^2; q^2)_{\infty} (zq; q)_{\infty}}{(z^2 q^2; q^2)_{\infty} (zq^2; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(-zq; q)_n (z; q^2)_n}{(q; q)_n (z^2 q^2; q^2)_n} q^n, \\ \sum_{n=0}^{\infty} \frac{(z; q^2)_{n+1} (q; q^2)_n}{(-zq; q)_{2n+1}} z^n &= (q)_{\infty} (zq; q^2)_{\infty} \sum_{n=0}^{\infty} \frac{(z; q^2)_n}{(zq; q)_n (q)_n} q^n. \end{aligned}$$

Equation (4.8) follows from (4.10) and (4.3). \square

Now we are ready to complete the proof of (1.16). As usual we prove that coefficient of z^k on both sides agrees for each k . Let

$$(4.11) \quad \text{LHS}(1.16) = L(z) = \sum_{k=-\infty}^{\infty} \ell_k(q) z^k,$$

$$(4.12) \quad \text{RHS}(1.16) = R(z) = \sum_{k=-\infty}^{\infty} r_k(q) z^k.$$

We see that

$$L(z) = z L(z^{-1}), \quad R(z) = z R(z^{-1}),$$

so that

$$\ell_k(q) = \ell_{1-k}(q), \quad r_k(q) = r_{1-k}(q),$$

for all k . Therefore we may assume that $k \geq 0$. We let $a = \rho^{-1}$, $b = q$, $c = -1$, $d = zq$, $e = z^{-1}q$ in [21, (III.10)]:

$$(4.13) \quad {}_3\phi_2 \left(\begin{matrix} \rho^{-1}, & q, & -1; & q, & -\rho q \\ & zq, & z^{-1}q & \end{matrix} \right) = \frac{(q, -\rho q, -q; q)_\infty}{(zq, z^{-1}q, -\rho q; q)_\infty} {}_3\phi_2 \left(\begin{matrix} z, & z^{-1}, & -\rho q; & q, & q \\ & \rho q, & -q & \end{matrix} \right).$$

We let $\rho \rightarrow 0^+$ and multiply both sides by $(q)_\infty(zq)_\infty(z^{-1}q)_\infty$ to find that

$$(4.14) \quad (1+z)(zq)_\infty(z^{-1}q)_\infty(q)_\infty \sum_{n=0}^{\infty} \frac{(-1)_n q^{\frac{1}{2}n(n+1)}}{(zq)_n(z^{-1}q)_n} = (1+z)(q^2; q^2)_\infty(q)_\infty \sum_{n=0}^{\infty} \frac{(z)_n(z^{-1})_n}{(q^2; q^2)_n} q^n.$$

From (4.1) and (4.14) we have

$$\begin{aligned} [z^k]L(z) &= (q)_\infty(q^2; q^2)_\infty \sum_{n=k-1}^{\infty} \frac{(-1)^{k+1}(q; q)_{2n}(1 - q^{2k-1})q^{n+\frac{1}{2}k(k-3)+1}}{(q)_{n+k}(q)_{n-k+1}(q^2; q^2)_n} \\ &= (q)_\infty(q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{(-1)^{k+1}(q; q^2)_{n+k-1}(1 - q^{2k-1})q^{n+\frac{1}{2}k(k-1)}}{(q)_{n+2k-1}(q)_{n-k+1}} \\ &= (q)_\infty(q^2; q^2)_\infty \frac{(q; q^2)_\infty(q^{2k}; q)_\infty}{(q^{2k-1}; q^2)_\infty(q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^{k+1}(q^{2k-1}; q^2)_n(1 - q^{2k-1})q^{n+\frac{1}{2}k(k-1)}}{(q^{2k}; q)_n(q)_n} \\ &= (q)_\infty(q^{2k}; q^2)_\infty \sum_{n=0}^{\infty} \frac{(-1)^{k+1}(q^{2k-1}; q^2)_n q^{n+\frac{1}{2}k(k-1)}}{(q^{2k}; q)_n(q)_n} \\ &= (-1)^{k+1} q^{\frac{1}{2}k(k-1)} + 2 \sum_{m=1}^{\infty} (-1)^{m+k+1} q^{m^2+(2k-1)m+\frac{1}{2}k(k-1)} \\ &\quad \text{(by letting } z = q^{2k-1} \text{ in (4.3))} \\ &= [z^k]R(z), \end{aligned}$$

as required. This completes the proof of (1.16).

We describe Milne's [33] bijective proof that

$$(4.15) \quad \sum_{n=0}^{\infty} \sum_{m=-[n/2]}^{[n/2]} (-1)^{n+m} q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n} = \sum_{n=0}^{\infty} \sum_{m=-[n/3]}^{[n/3]} (-1)^n q^{\frac{1}{2}(n^2-8m^2)+\frac{1}{2}n}.$$

Let

$$\begin{aligned} \mathcal{S}_1 &= \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : n \geq 2|m|\}, \\ \mathcal{S}_2 &= \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : n \geq 3|m|\}. \end{aligned}$$

Define

$$T : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$$

by

$$T(m, n) = \begin{cases} (\frac{1}{2}m, n) & \text{if } m \geq 0 \text{ is even,} \\ (n - \frac{3}{2}m + \frac{1}{2}, 3n - 4m + 1) & \text{if } m \geq 1 \text{ is odd,} \end{cases}$$

and

$$T(-m, n) = (-m_1, n_1) \quad \text{if } T(m, n) = (m_1, n_1).$$

Milne proved (4.15) by showing that T is a bijection that satisfies

$$Q_2(T(m, n)) = Q_1(m, n),$$

where

$$\begin{aligned} Q_1(m, n) &= \frac{1}{2}n^2 - m^2 + \frac{1}{2}n, \\ Q_2(m, n) &= \frac{1}{2}n^2 - 4m^2 + \frac{1}{2}n. \end{aligned}$$

The same bijection proves that the right sides of (1.16) and (1.17) are equal, since it is not difficult to show that the transformation T also satisfies

$$\begin{aligned} L_{2,1}(T(m, n)) &= \begin{cases} L_{1,1}(m, n) & \text{if } m \text{ is even} \\ L_{1,2}(m, n) & \text{if } m \text{ is odd,} \end{cases} \\ L_{2,2}(T(m, n)) &= \begin{cases} L_{1,2}(m, n) & \text{if } m \text{ is even} \\ L_{1,1}(m, n) & \text{if } m \text{ is odd,} \end{cases} \end{aligned}$$

where

$$\begin{aligned} L_{1,1}(m, n) &= n - 2|m| + 1, \\ L_{1,2}(m, n) &= 2|m| - n, \\ L_{2,1}(m, n) &= n - 4|m| + 1, \\ L_{2,2}(m, n) &= 4|m| - n, \end{aligned}$$

and

$$S_2(T(m, n)) \equiv S_1(m, n) \pmod{2},$$

where

$$\begin{aligned} S_1(m, n) &= m + n, \\ S_2(m, n) &= n. \end{aligned}$$

This completes the proof of (1.16) and (1.17).

5. A TWO-VARIABLE HECKE-ROGERS IDENTITY FOR THE $M2$ -RANK FUNCTION

In this section we prove (1.18). First we need a result similar to Proposition 4.2.

Proposition 5.1.

$$(5.1) \quad \frac{(zq; q)_\infty}{(-q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-zq; q)_{2n} (-1)^n z^n q^n}{(z^2 q^2; q^2)_n (q^2; q^2)_n} = \sum_{m=0}^{\infty} (-1)^m z^m q^{\frac{1}{2}m(m+1)}.$$

Proof. We apply Heine's transformation [21, (III.2)] with $a = -zq^2$, $b = -zq$, $c = z^2 q^2$, $q \mapsto q^2$ and $z \mapsto q$ to obtain

$$(5.2) \quad \frac{(zq; q)_\infty}{(-q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-zq; q)_{2n} (-1)^n z^n q^n}{(z^2 q^2; q^2)_n (q^2; q^2)_n} = \sum_{n=0}^{\infty} \frac{(-zq; q^2)_n (-zq)^n}{(-zq^2; q^2)_n},$$

after some simplification. The result (5.1) now follows from

$$(5.3) \quad \sum_{n=0}^{\infty} \frac{(-zq; q^2)_n (-zq)^n}{(-zq^2; q^2)_n} = \sum_{n=0}^{\infty} (-1)^n z^n q^{\frac{1}{2}n(n+1)},$$

which is Entry 9.3.1 in Ramanujan's Lost Notebook [7, Eq.(9.3.1), p.227]. \square

Now we are ready to complete the proof of (1.18). It is clear that the coefficient of z^k on the left side of (1.18) equals the coefficient of z^{-k} . We see that the same is true for the right side after we rewrite it as

$$\sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n+1)} + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} (-1)^n (z^{n-m} + z^{m-n}) q^{\frac{1}{2}(2n^2-m^2) + \frac{1}{2}(2n-m)}.$$

Thus we may assume that $k \geq 0$. We let $q \rightarrow q^2$, $a = \rho^{-1}$, $b = q$, $c = q^2$, $d = zq^2$, $e = z^{-1}q^2$ in [21, (III.10)]:

$$(5.4) \quad {}_3\phi_2 \left(\begin{matrix} \rho^{-1}, & q, & q^2; & q^2, & \rho q \\ & zq^2, & z^{-1}q^2 & \end{matrix} \right) = \frac{(q, \rho q^3, q; q^2)_\infty}{(zq^2, z^{-1}q^2, \rho q; q^2)_\infty} {}_3\phi_2 \left(\begin{matrix} zq, & z^{-1}q, & \rho q; & q^2, & q \\ & \rho q^3, & q & \end{matrix} \right).$$

We let $\rho \rightarrow 0^+$ and multiply both sides by $(q^2; q^2)_\infty (zq^2; q^2)_\infty (z^{-1}q^2; q^2)_\infty$ to find that

$$(5.5) \quad (zq^2; q^2)_\infty (z^{-1}q^2; q^2)_\infty (q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{n^2}}{(zq^2; q^2)_n (z^{-1}q^2; q^2)_n} = \frac{(q)_\infty}{(-q)_\infty} \sum_{n=0}^{\infty} \frac{(zq; q^2)_n (z^{-1}q; q^2)_n}{(q; q^2)_n (q^2; q^2)_n} q^n.$$

In (4.2) we let $q \rightarrow q^2$, $z \rightarrow zq$ to obtain

$$(5.6) \quad (zq; q^2)_n (z^{-1}q; q^2)_n = \sum_{k=-n}^n (-1)^k z^k q^{k^2} \left[\begin{matrix} 2n \\ n+k \end{matrix} \right]_{q^2}.$$

From (5.6) and (5.5) we have

$$\begin{aligned}
[z^k]\text{LHS}(1.18) &= \frac{(q)_\infty}{(-q)_\infty} \sum_{n=k}^{\infty} \frac{(-1)^k (q^2; q^2)_{2n} q^{n+k^2}}{(q^2; q^2)_{n+k} (q^2; q^2)_{n-k} (q; q)_{2n}} \\
&= \frac{(q)_\infty}{(-q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^k (-q; q)_{2n+2k} q^{n+k^2+k}}{(q^2; q^2)_{n+2k} (q^2; q^2)_n} \\
&= \frac{(q^{4k+2}; q^2)_\infty}{(-q; q)_\infty (-q^{2k+1}; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^k (-q^{2k+1}; q)_{2n} q^{n+k^2+k}}{(q^{4k+2}; q^2)_n (q^2; q^2)_n} \\
&= \frac{(q^{2k+1}; q)_\infty}{(-q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^k (-q^{2k+1}; q)_{2n} q^{n+k^2+k}}{(q^{4k+2}; q^2)_n (q^2; q^2)_n} \\
&= \sum_{m=0}^{\infty} (-1)^{m+k} q^{\frac{1}{2}m(m+1) + (2m+1)k + k^2} \\
&\quad \text{(by letting } z = q^{2k} \text{ in (5.1))} \\
&= [z^k]\text{RHS}(1.18)
\end{aligned}$$

as required. This completes the proof of (1.18).

6. TWO-VARIABLE HECKE-ROGERS IDENTITIES FOR OTHER SPT-CRANK FUNCTIONS

Let $\overline{\text{S}}(z, q)$ be the generating function for the spt-crank function for overpartitions [20]. Then

$$(6.1) \quad \overline{\text{S}}(z, q) = \sum_{n=1}^{\infty} \frac{q^n (q^{2n+2}; q^2)_\infty}{(zq^n; q)_\infty (z^{-1}q^n; q)_\infty}$$

$$(6.2) \quad = \sum_{n=1}^{\infty} \sum_m N_{\overline{\text{S}}}(m, n) z^m q^n.$$

We note that

$$\overline{\text{S}}(1, q) = \sum_{n=1}^{\infty} \frac{q^n (-q^{n+1}; q)_\infty}{(1 - q^n)^2 (q^{n+1}; q)_\infty} = \sum_{n=1}^{\infty} \overline{\text{spt}}(n) q^n,$$

where $\overline{\text{spt}}(n)$ is the number of smallest parts in the overpartitions of n , where we are using the convention that the smallest part is not overlined. The spt-crank function for overpartitions can be written in terms of the rank and crank functions for overpartitions.

$$(6.3) \quad \overline{\text{S}}(z, q) = \frac{1}{(1-z)(1-z^{-1})} \sum_{n=1}^{\infty} \sum_m (\overline{N}(m, n) - \overline{M}(m, n)) z^m q^n,$$

where

$$(6.4) \quad \sum_{n=0}^{\infty} \sum_m \overline{N}(m, n) z^m q^n = \sum_{n=0}^{\infty} \frac{(-1; q)_n q^{n(n+1)/2}}{(zq; q)_n (z^{-1}q; q)_n},$$

and

$$(6.5) \quad \sum_{n=0}^{\infty} \sum_m \overline{M}(m, n) z^m q^n = \frac{(-q; q)_{\infty} (q; q)_{\infty}}{(zq; q)_{\infty} (z^{-1}q; q)_{\infty}}.$$

We find the following analog of Theorem 2.1.

Theorem 6.1.

$$(6.6) \quad (1+z)(z)_{\infty} (z^{-1})_{\infty} (q)_{\infty} \overline{S}(z, q) \\ = \sum_{n=0}^{\infty} \sum_{m=-[n/2]}^{[n/2]} (-1)^{m+n} (1 - z^{n-2|m|+1}) (1 - z^{n-2|m|}) z^{2|m|-n} q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n}.$$

Proof. Equation (6.6) follows in a straightforward manner from (1.2), (1.16), (6.3), (6.4) and (6.5). \square

If we divide both sides of (6.6) by $(1-z)(1-z^{-1})$ and let $z \rightarrow 1$ we obtain

Corollary 6.2.

$$(6.7) \quad \prod_{n=1}^{\infty} (1 - q^n)^3 \sum_{n=1}^{\infty} \overline{\text{spt}}(n) q^n = \sum_{n=0}^{\infty} \sum_{m=-[n/2]}^{[n/2]} (-1)^{m+n+1} \binom{n-2|m|+1}{2} q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n}.$$

Let $S2(z, q)$ be the generating function for the spt-crank function for partitions with distinct odd parts and smallest part even [20]. Then

$$(6.8) \quad S2(z, q) = \sum_{n=1}^{\infty} \frac{q^{2n} (q^{2n+2}; q^2)_{\infty} (-q^{2n+1}; q^2)_{\infty}}{(zq^{2n}; q^2)_{\infty} (z^{-1}q^{2n}; q^2)_{\infty}} \\ = \sum_{n=1}^{\infty} \sum_m N_{S2}(m, n) z^m q^n.$$

We note that

$$S2(1, q) = \sum_{n=1}^{\infty} \frac{q^{2n} (-q^{2n+1}; q^2)_{\infty}}{(1 - q^{2n})^2 (q^{2n+2}; q^2)_{\infty}} = \sum_{n=1}^{\infty} M2\text{spt}(n) q^n,$$

where $M2\text{spt}(n)$ is the number of smallest parts in the partitions of n without repeated odd parts and with smallest part even. This function was studied by Ahlgren,

Bringmann and Lovejoy [1]. Again we find this spt-crank function be written in terms of the relevant rank and crank functions.

$$(6.9) \quad S2(z, q) = \frac{1}{(1-z)(1-z^{-1})} \sum_{n=1}^{\infty} \sum_m (N2(m, n) - M2(m, n)) z^m q^n,$$

where

$$(6.10) \quad \sum_{n=0}^{\infty} \sum_m N2(m, n) z^m q^n = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(zq^2; q^2)_n (z^{-1}q^2; q^2)_n},$$

and

$$(6.11) \quad \sum_{n=0}^{\infty} \sum_m M2(m, n) z^m q^n = \frac{(-q; q^2)_{\infty} (q^2; q^2)_{\infty}}{(zq^2; q^2)_{\infty} (z^{-1}q^2; q^2)_{\infty}}.$$

We find the following analog of Theorem 2.1.

Theorem 6.3.

$$(6.12) \quad (z; q^2)_{\infty} (z^{-1}; q^2)_{\infty} (q^2; q^2)_{\infty} S2(z, -q) \\ = \sum_{n=0}^{\infty} \sum_{m=0}^n (-1)^n (1 - z^{n-m})^2 z^{m-n} q^{\frac{1}{2}(2n^2 - m^2) + \frac{1}{2}(2n-m)}.$$

Proof. From (6.9), (6.10), (6.11) we have

$$(6.13) \quad (z; q^2)_{\infty} (z^{-1}; q^2)_{\infty} (q^2; q^2)_{\infty} S2(z, -q) \\ = (zq^2; q^2)_{\infty} (z^{-1}q^2; q^2)_{\infty} (q^2; q^2)_{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{n^2}}{(zq^2; q^2)_n (z^{-1}q^2; q^2)_n} - (q; q)_{\infty} (q^2; q^2)_{\infty}.$$

We note that

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=1}^n (-1)^n z^{n-m+1} q^{\frac{1}{2}(2n^2 - m^2) + \frac{1}{2}(2n+m)} \\ &= \sum_{n=0}^{\infty} \sum_{m=1}^n (-1)^n z^{n-(m-1)} q^{\frac{1}{2}(2n^2 - (m-1)^2) + \frac{1}{2}(2n-(m-1))} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} (-1)^n z^{n-m} q^{\frac{1}{2}(2n^2 - m^2) + \frac{1}{2}(2n-m)}. \end{aligned}$$

Thus from (1.4), (1.18) and (6.13) we have

$$(6.14) \quad (z; q^2)_{\infty} (z^{-1}; q^2)_{\infty} (q^2; q^2)_{\infty} S2(z, -q)$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{m=0}^n (-1)^n (z^{m-n} + z^{n-m} - 2) q^{\frac{1}{2}(2n^2-m^2) + \frac{1}{2}(2n-m)} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^n (-1)^n (1 - z^{n-m})^2 z^{m-n} q^{\frac{1}{2}(2n^2-m^2) + \frac{1}{2}(2n-m)},
\end{aligned}$$

which is the result. \square

If we divide both sides of (6.12) by $(1-z)(1-z^{-1})$ and let $z \rightarrow 1$ we obtain

Corollary 6.4.

$$(6.15) \quad \prod_{n=1}^{\infty} (1 - q^{2n})^3 \sum_{n=1}^{\infty} (-1)^n \text{M2spt}(n) q^n = \sum_{n=1}^{\infty} \sum_{m=0}^n (-1)^{n+1} (n-m)^2 q^{\frac{1}{2}(2n^2-m^2) + \frac{1}{2}(2n-m)}.$$

Define the function $\beta(n)$ by

$$(6.16) \quad \sum_{n=1}^{\infty} \beta(n) q^n = \prod_{n=1}^{\infty} (1 - q^{16n})^3 \sum_{n=1}^{\infty} (-1)^n \text{M2spt}(n) q^{8n+1},$$

so that $\beta(n) = 0$ if n is not a positive integer congruent to 1 (mod 8). We have

Corollary 6.5. *Suppose $\ell \equiv \pm 3 \pmod{8}$ is prime. Then*

$$\begin{aligned}
\beta(\ell n) + \ell^2 \beta(n/\ell) &= 0, & \text{if } \ell \equiv 3 \pmod{8}, \\
\beta(\ell n) - \ell^2 \beta(n/\ell) &= 0, & \text{if } \ell \equiv 5 \pmod{8}.
\end{aligned}$$

Proof. From (6.15), (6.16) we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \beta(n) q^n &= \sum_{n=0}^{\infty} \sum_{j=0}^n (-1)^{n+1} (n-m)^2 q^{2(2n+1)^2 - (2m+1)^2} \\
&= - \sum_{n=1}^{\infty} \sum_{m=1}^n \left(\frac{n-m}{2} \right)^2 \left(\frac{-4}{n} \right) \left(\frac{4}{m} \right) q^{2n^2 - m^2}.
\end{aligned}$$

Suppose ℓ is prime and $\ell \equiv \pm 3 \pmod{8}$. Then we observe that

$$2n^2 - m^2 \equiv 0 \pmod{\ell} \quad \text{if and only if} \quad n \equiv m \equiv 0 \pmod{\ell},$$

since 2 is quadratic nonresidue mod ℓ . Hence

$$\beta(\ell n) = \left(\frac{-4}{\ell} \right) \ell^2 \beta(n/\ell),$$

which gives the result. \square

7. CONCLUDING REMARKS

There are other two-variable Hecke-Rogers identities in the literature. Andrews [4] proved the following identity

$$(7.1) \quad \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-zq^n)(1-z^{-1}q^{n-1})} = \sum_{n=0}^{\infty} \sum_{m=-n}^n (-1)^{m+n} z^m q^{\frac{1}{2}(n^2-m^2) + \frac{1}{2}(n+m)},$$

where $|q| < 1$ and $1 < |z| < |q|^{-1}$. Andrews used this identity to show how elementary q -series techniques could be used to prove identities such as (1.1)–(1.4). Hickerson and Mortenson [26] studied the function

$$(7.2) \quad f_{a,b,c}(x, y, q) := \sum_{\text{sgn}(r)=\text{sgn}(s)} \text{sgn}(r) (-1)^{r+s} x^r y^s q^{a\binom{r}{2} + brs + c\binom{s}{2}}.$$

They found a general identity for this function in terms of Apell-Lerch sums and theta functions. Their formula not only proves the known Hecke-Rogers identities such as (1.1)–(1.4) but also leads to new straightforward proofs of many of the classical mock theta function identities, including a new proof of the mock theta conjectures [8]. It would be interesting to determine whether the methods and results of Hickerson and Mortenson [26] can be used to give an alternative proof of our main result Theorem 1.1. It would also be interesting to see whether the theory of mock Jacobi forms [16], [17], [38] could be developed to derive these results.

Our proof of our rank function result (1.15) depends on first proving the spt-crank result (2.7) in Theorem 2.1. The proof of (2.7) utilizes the method of Bailey pairs. It is possible to give a direct proof of (1.15) using Bailey pair technology. We leave this to the interested reader. We were unable to find a proof of the other rank-type function results (1.16)–(1.18) by the method of Bailey pairs.

Lovejoy [30] has found a number of identities that give certain q -hypergeometric sums in terms of two-variable Hecke-Rogers type series using the method of Bailey pairs. Lovejoy [31] has also found families of q -hypergeometric mock theta multisums in terms of Hecke-Rogers type double series.

Mortenson [35] has utilized Lovejoy's [30] method also obtain some similar identities. We give three examples. From results in [35, Section 4.3] it can be shown that

$$(7.3) \quad (q)_{\infty} (1+z^{-1}) \sum_{n=0}^{\infty} \frac{(-zq; q^2)_n (-z^{-1}q; q^2)_n q^{2n}}{(q; q^2)_{n+1}} \\ = \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor n/3 \rfloor} (z^{n-3m} + z^{3m-n-1}) q^{(n^2-3m^2) + (2n-m)} (1 - q^{4n-4m+6}).$$

Dividing both sides by $(1 + z^{-1})$ and letting $z \rightarrow -1$ yields

$$(7.4) \quad (q)_\infty \sum_{n=0}^{\infty} \frac{(q; q^2)_n q^{2n}}{(1 - q^{2n+1})} \\ = \sum_{n=0}^{\infty} \sum_{m=0}^{[n/3]} (-1)^{m+n} (2n - 6m + 1) q^{(n^2 - 3m^2) + (2n - m)} (1 - q^{4n - 4m + 6}).$$

Similarly, from results in [35, Section 4.4] we find that

$$(7.5) \quad (-q; q^4)_\infty (-q^3; q^4)_\infty (q^4; q^4)_\infty (1 + z^{-1}) \sum_{n=0}^{\infty} \frac{(zq; q^2)_n (z^{-1}q; q^2)_n q^{2n}}{(q; q)_{2n+1}} \\ = \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} (-1)^m (z^m + z^{-m-1}) q^{\frac{1}{2}(n^2 - 2m^2) + \frac{1}{2}(3n - 2m)},$$

and

$$(7.6) \quad \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{2n}}{(-q^2; q^2)_n (1 + q^{2n+1})} \\ = \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} (2m + 1) q^{\frac{1}{2}(n^2 - 2m^2) + \frac{1}{2}(3n - 2m)}.$$

Also, from results in [35, Section 4.6] we find that

$$(7.7) \quad (q; q^2)_\infty (q; q)_\infty (1 + z^{-1}) \sum_{n=0}^{\infty} \frac{(-zq; q)_n (-z^{-1}q; q^2)_n q^{n+1}}{(q; q^2)_n} \\ = \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} (-1)^m (z^{n-2m} + z^{-n+2m-1}) q^{\frac{1}{2}(n^2 - 2m^2) + \frac{1}{2}(3n - 2m)},$$

and

$$(7.8) \quad \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \sum_{n=0}^{\infty} \frac{(q; q)_n^2 q^n}{(q; q^2)_{n+1}} \\ = \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} (-1)^{m+n} (2n - 4m + 1) q^{\frac{1}{2}(n^2 - 2m^2) + \frac{1}{2}(3n - 2m)}.$$

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